Mathematical Proof of the Haversine Distance Formula

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Introduction

The Haversine distance formula provides a method to calculate the shortest distance between two points on the surface of a sphere, given their longitudes and latitudes. This formula is widely used in navigation and geodesy.

Let the two points be represented by their geographic coordinates:

- Point 1: (ϕ_1, λ_1) where ϕ_1 is the latitude and λ_1 is the longitude of point 1.
- Point 2: (ϕ_2, λ_2) where ϕ_2 is the latitude and λ_2 is the longitude of point 2.

We aim to derive the formula for the great-circle distance between these two points.

Mathematical Derivation

1. **Convert the latitudes and longitudes to radians**: The coordinates should be in radians, so we convert the degrees into radians:

$$\phi_1 = \text{latitude of point 1 in radians}, \quad \lambda_1 = \text{longitude of point 1 in radians}$$

$$\phi_2 = \text{latitude of point 2 in radians}, \quad \lambda_2 = \text{longitude of point 2 in radians}$$

2. **Apply the spherical law of cosines**: The spherical law of cosines gives the central angle $\Delta\sigma$ between the two points on the sphere:

$$\cos(\Delta\sigma) = \sin(\phi_1)\sin(\phi_2) + \cos(\phi_1)\cos(\phi_2)\cos(\lambda_2 - \lambda_1)$$

where $\Delta\sigma$ is the central angle between the two points.

3. **Define the Haversine function**: The Haversine function is defined as:

$$\text{hav}(\theta) = \sin^2\left(\frac{\theta}{2}\right)$$

Now, we apply the Haversine formula for calculating the central angle $\Delta\sigma$:

$$\text{hav}(\Delta\sigma) = \sin^2\left(\frac{\Delta\sigma}{2}\right) = \sin^2\left(\frac{\phi_2 - \phi_1}{2}\right) + \cos(\phi_1)\cos(\phi_2)\sin^2\left(\frac{\lambda_2 - \lambda_1}{2}\right)$$

4. **Solve for the distance**: Now, we can compute the distance d on the surface of a sphere of radius R (such as the Earth with radius approximately 6371 km):

$$d = 2R \cdot \arcsin\left(\sqrt{\text{hav}(\Delta\sigma)}\right)$$

Substituting for $\text{hav}(\Delta\sigma)$:

$$d = 2R \cdot \arcsin\left(\sqrt{\sin^2\left(\frac{\phi_2 - \phi_1}{2}\right) + \cos(\phi_1)\cos(\phi_2)\sin^2\left(\frac{\lambda_2 - \lambda_1}{2}\right)}\right)$$

Conclusion

Thus, the Haversine formula provides the great-circle distance between two points on a sphere, considering their latitudes and longitudes. The distance d is given by:

$$d = 2R \cdot \arcsin\left(\sqrt{\sin^2\left(\frac{\phi_2 - \phi_1}{2}\right) + \cos(\phi_1)\cos(\phi_2)\sin^2\left(\frac{\lambda_2 - \lambda_1}{2}\right)}\right)$$